New (G'/G) -expansion method and its applications to nonlinear PDE

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Abstract

In this paper, the new (G'/G) -expansion method is proposed for constructing more general exact solutions of nonlinear evolution equation with the aid of symbolic computation.By using this method many new and more general exact solutions have been obtained.To illustrate the novelty and advantage of the proposed method, we solve the Zakharov-Kuznetsov-Benjamin-Bona-Mahony (ZKBBM) equation. Abundant exact travelling wave solutions of these equations are obtained, which include the exponential function solutions, the hyperbolic function solutions and the trigonometric function solutions. Also it is shown that the proposed method is efficient for solving nonlinear evolution equations in mathematical physics and in engineering.

Keywords:(G'/G) -expansion method;ZKBBM equation;exact solutions;
MSC-AMS: 35Q53, 35Q51,

1 Introduction

In recent years, because of the wide applications of soliton theory in natural science, it is important to seek explicit exact solutions of nonlinear partial differential equations (NLPDEs).Many powerful methods for constructing exact solutions of nonlinear evolution equations have been established and developed, such as the inverse scattering transform [1], the Backlund transform [2], the Hirota's bilinear operators [3],the tanh-coth function expansion[4], the Jacobi elliptic function expansion [5], the F-expansion [6], the sub-ODE method [7], the homogeneous balance method [8], the sine-cosine method [9],the exp-function expansion method[10] and so on. But there is no unified method that can be used to deal with all types of nonlinear evolution equations.
Recently, Wang introduced the(G'/G)-expansion method.This method is widely used for constructing exact solutions of various NLEEs. Applications of the (G'/G)-expansion method

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can be found in other articles for better understanding[11][13].
The present paper is motivated by the desire to use the improved $(G^{'}/G)$ -expansion method to construct a series of some types of exact solutions.We will get more interaction solutions of the nonlinear Zakharov-Kuznetsov- Benjamin-Bona-Mahony (ZKBBM) equation,which are very important nonlinear evolution equations in the mathematical physics and have been paid attention by many researchers.

2 Summary of the expansion method

The new auxiliary ordinary differential equation is expressed as follows:

$$GG^{''} = AG^2 + BGG^{'} + C(G^{'})^2 \tag{1}$$

where the prime denotes derivative with respect to ξ. A, B, C are real parameters.$F(\xi)$ is

$$F(\xi) = \frac{G^{'}(\xi)}{G(\xi)} \tag{2}$$

Using the general solutions of Eq. (1), with the help of Maple we have the following four solutions of Eq. (2):
(i). when $B \neq 0$ and $\Delta_1 = B^2 + 4A - 4AC \geq 0$, then

$$F(\xi) = \frac{B}{2(1-C)} + \frac{B\sqrt{\Delta_1}}{2(1-C)} \frac{c_1 \exp^{\frac{\sqrt{\Delta_1}}{2}\xi} + c_2 \exp^{-\frac{\sqrt{\Delta_1}}{2}\xi}}{c_1 \exp^{\frac{\sqrt{\Delta_1}}{2}\xi} - c_2 \exp^{-\frac{\sqrt{\Delta_1}}{2}\xi}} \tag{3}$$

(ii). when $B \neq 0$ and $\Delta_1 = B^2 + 4A - 4AC < 0$, then

$$F(\xi) = \frac{B}{2(1-C)} + \frac{B\sqrt{-\Delta_1}}{2(1-C)} \frac{ic_1 \cos(\frac{\sqrt{-\Delta_1}}{2}\xi) - c_2 \sin(-\frac{\sqrt{-\Delta_1}}{2}\xi)}{ic_1 \sin(\frac{\sqrt{-\Delta_1}}{2}\xi) + c_2 \cos(-\frac{\sqrt{-\Delta_1}}{2}\xi)} \tag{4}$$

(iii).when $B = 0$ and $\Delta_2 = A(C-1) \geq 0$, then

$$F(\xi) = \frac{\sqrt{\Delta_2}}{(1-C)} \frac{c_1 \cos(\sqrt{\Delta_2}\xi) + c_2 \sin(\sqrt{\Delta_2}\xi)}{c_1 \sin(\sqrt{\Delta_2}\xi) - c_2 \cos(\sqrt{\Delta_2}\xi)} \tag{5}$$

(iv). when $B = 0$ and $\Delta_2 = A(C-1) < 0$, then

$$F(\xi) = \frac{\sqrt{-\Delta_2}}{(1-C)} \frac{ic_1 \cosh(\sqrt{-\Delta_2}\xi) - c_2 \sinh(\sqrt{-\Delta_2}\xi)}{ic_1 \sinh(\sqrt{-\Delta_2}\xi) - c_2 \cosh(\sqrt{-\Delta_2}\xi)} \tag{6}$$

where $\xi = x - \omega t$,ω is wave velocity,A, B, C and c_1, c_2 are real parameters.
Suppose that we have a NLEE for u(x, t) in the form

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \dots) = 0 \tag{7}$$

where P is a polynomial in its arguments, which includes nonlinear terms and the highest order derivatives.

Next,the main steps of this method are given as follows:

Step 1.The transformation $u(x,t) = u(\xi)$, $\xi = x - \omega t$ reduces Eq. (7) to the ordinary differential equation (ODE)

$$H(u, u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}, \dots) = 0 \tag{8}$$

Step 2.We assume that the solution of Eq. (8) is of the form

$$u(\xi) = \sum_{i=-m}^{m} a_i(d + F(\xi))^i \tag{9}$$

where $F(\xi)$ satisfy the new auxiliary ordinary differential Eq.(1),and $\omega, d, a_i(i = -m, \dots, m)$ can be determined later.We can determine the positive integer n by balancing the highest nonlinear terms and the highest partial derivative terms in the given system equations.

Step 3.Substituting Eq. (9) along with (1) and (2)into Eq. (8) and using Maple yields a system of equations of $F^i(\xi)$, setting the coefficients of $F^i(\xi)$ in the obtained system of equations to zero, we can deduce the algebraic polynomials with the respect unknowns $\omega, d, a_i(i = -m, \dots, m)$ namely.

Step 4.Solving the over-determined system of algebraic equations by using the symbolic computation as Maple , we obtain expressions for $\omega, d, a_i(i = -m, \dots, m)$.

Step 5. Since the general solutions of (1) have been well known for us, then substituting $\omega, d, a_i(i = -m, \dots, m)$ and the general solutions (3)-(6) into (2), we have more exact solutions of the non-linear partial differential Eq. (7).

3 Travelling wave solitons for ZKBBM equation

In this section,we apply this method to construct the exact interaction soliton solutions of the ZKBBM equation

$$u_t + u_x - 2auu_x - bu_{xxxt} = 0 \tag{10}$$

The transformation $u(x,t) = u(\xi)$,$\xi = x + Vt$ reduces Eq. (10) to the ordinary differential equation (ODE):

$$(1 + V)u^{'} - 2auu^{'} - bVu^{'''} = 0 \tag{11}$$

We can determine the positive integer n by balancing $uu^{'}$ and $u^{'''}$ in the given system equations.So we can suppose that Eq. (11) has the following ansatz:

$$u(\xi) = \frac{a_{-2}}{(d + F(\xi))^2} + \frac{a_{-1}}{d + F(\xi)} + a_0 + a_1(d + F(\xi)) + a_2(d + F(\xi))^2 \tag{12}$$

Substituting Eq. (12) along with (1) and (2)into Eq. (11) and using Maple yields a system of equations of $F^i(\xi)$, setting the coefficients of $F^i(\xi)(i = 0, 1, 2, \dots)$ in the obtained system of equations to zero, we can deduce the set of algebraic polynomials with the respect unknowns $V, d, a_i(i = -m, \dots, m)$ namely.Solving the over-determined system of algebraic equations by using the symbolic computation as Maple , we obtain expressions for $V, d, a_i(i = -m, \dots, m)$.

Case 1.

$$d = \frac{1}{2}\frac{B}{C-1}, V = V, a_1 = 0, a_2 = 0$$

$$a_{-1} = 0, a_0 = -\frac{1}{2}\frac{(8bVAC - 8bVA - 2bVB^2 - 1 - V)}{a}$$

$$a_{-2} = -\frac{3}{8}\frac{bV(-8AB^2C + 8AB^2 + B^4 + 16A^2C^2 - 32A^2C + 16A^2)}{a(C^2 - 2C + 1)} \tag{3.1}$$

where A, B, C, V, a, b are arbitrary constants,and $a \neq 0, C \neq 1$.

Case 2.

$$d = \frac{2A}{B}, V = V, a_1 = 0, a_2 = 0$$

$$a_{-2} = -\frac{6bVA^2(-8AB^2C + 8AB^2 + B^4 + 16A^2C^2 - 32A^2C + 16A^2)}{aB^4}$$

$$a_{-1} = \frac{6bVA(-8AB^2C + 8AB^2 + B^4 + 16A^2C^2 - 32A^2C + 16A^2)}{aB^3}$$

$$a_0 = -\frac{1}{2}\frac{48bVA^2C^2 - 16bVACB^2 - 96bVA^2C + 16bVAB^2 - B^2 - VB^2 + 48bVA^2 + bVB^4}{B^2a}$$

where A, B, C, V, a, b are arbitrary constants,and $B \neq 0, a \neq 0$.

Case 3.

$$d = d, V = V, a_{-2} = 0, a_{-1} = 0$$

$$a_1 = -\frac{6bV(-2dC^2 + 4Cd + BC - 2d - B)}{a}, a_2 = -\frac{6bV(C^2 - 2C + 1)}{a}$$

$$a_0 = -\frac{1}{2}\frac{12bVC^2d^2 + 8bVAC - 24bVCd^2 - 12bVBCd - 1 + 12bVd^2 + bVB^2 + 12bVBd - V - 8bVA}{a}$$

where A, B, C, V, d, a, b are arbitrary constants,and$a \neq 0$.

Case 4.

$$d = \frac{1}{2}\frac{B}{C-1}, V = V, a_{-1} = 0, a_1 = 0$$

$$a_0 = -\frac{1}{2}\frac{8bVAC - 8bVA - 2bVB^2 - 1 - V}{a}, a_2 = -\frac{6bV(C^2 - 2C + 1)}{a}$$

$$a_{-2} = -\frac{3}{8}\frac{bV(-8AB^2C + 8AB^2 + B^4 + 16A^2C^2 - 32A^2C + 16A^2)}{a(C^2 - 2C + 1)}$$

where A, B, C, V, a, b are arbitrary constants,and $a \neq 0, C \neq 1$.

Substituting those cases in (12),we obtain the following solutions of Eq. (11). These solutions are:

$$u_1(\xi) = -\frac{3}{8}\frac{bV(-8AB^2C+8AB^2+B^4+16A^2C^2-32A^2C+16A^2)}{a(C^2-2C+1)}\frac{1}{(\frac{1}{2}\frac{B}{C-1}+F(\xi))^2}$$

$$-\frac{1}{2}\frac{(8bVAC-8bVA-2bVB^2-1-V)}{a}$$

$$u_2(\xi) = -\frac{6bVA^2(-8AB^2C+8AB^2+B^4+16A^2C^2-32A^2C+16A^2)}{aB^4}\frac{1}{(\frac{2A}{B}+F(\xi))^2}$$

$$+\frac{6bVA(-8AB^2C+8AB^2+B^4+16A^2C^2-32A^2C+16A^2)}{aB^3}\frac{1}{\frac{2A}{B}+F(\xi)}-\frac{1}{2}$$

$$\frac{48bVA^2C^2-16bVACB^2-96bVA^2C+16bVAB^2-B^2-VB^2+48bVA^2+bVB^4}{B^2a}$$

$$u_3(\xi) = -\frac{1}{2}\frac{12bVC^2d^2+8bVAC-24bVCd^2-12bVBCd-1+12bVd^2+bVB^2+12bVBd-V-8bVA}{a}$$

$$-\frac{6bV(-2dC^2+4Cd+BC-2d-B)}{a}(d+F(\xi))-\frac{6bV(C^2-2C+1)}{a}(d+F(\xi))^2$$

$$u_4(\xi) = -\frac{3}{8}\frac{bV(-8AB^2C+8AB^2+B^4+16A^2C^2-32A^2C+16A^2)}{a(C^2-2C+1)}\frac{1}{(\frac{1}{2}\frac{B}{C-1}+F(\xi))^2}-\frac{1}{2}$$

$$\frac{8bVAC-8bVA-2bVB^2-1-V}{a}-\frac{6bV(C^2-2C+1)}{a}(\frac{1}{2}\frac{B}{C-1}+F(\xi))^2$$

where$\xi = x+Vt$.

According to (3)-(6), we obtain the following exponential function solutions, hyperbolic function solutions and triangular function solutions of Eq. (10). For example

(1).When $B \neq 0$ and $\Delta_1 = B^2+4A-4AC \geq 0$,then

$$u_{41}(x,t) = -\frac{3}{8}\frac{bV(-8AB^2C+8AB^2+B^4+16A^2C^2-32A^2C+16A^2}{a(C^2-2C+1)}$$

$$\frac{1}{(\frac{B\sqrt{\Delta_1}}{2(1-C)}\frac{c_1\exp^{\frac{\sqrt{\Delta_1}}{2}(x+Vt)}+c_2\exp^{-\frac{\sqrt{\Delta_1}}{2}(x+Vt)}}{c_1\exp^{\frac{\sqrt{\Delta_1}}{2}(x+Vt)}-c_2\exp^{-\frac{\sqrt{\Delta_1}}{2}(x+Vt)}})^2}$$

$$-\frac{1}{2}\frac{8bVAC-8bVA-2bVB^2-1-V}{a}-\frac{6bV(C^2-2C+1)}{a}$$

$$(\frac{B\sqrt{\Delta_1}}{2(1-C)}\frac{c_1\exp^{\frac{\sqrt{\Delta_1}}{2}\xi}+c_2\exp^{-\frac{\sqrt{\Delta_1}}{2}\xi}}{c_1\exp^{\frac{\sqrt{\Delta_1}}{2}\xi}-c_2\exp^{-\frac{\sqrt{\Delta_1}}{2}\xi}})^2$$

(2).When $B \neq 0$ and $\Delta_1 = B^2 + 4A - 4AC < 0$,then

$$u_{42}(x,t) = -\frac{3}{8}\frac{bV(-8AB^2C + 8AB^2 + B^4 + 16A^2C^2 - 32A^2C + 16A^2}{a(C^2 - 2C + 1)}$$

$$\frac{1}{(\frac{B\sqrt{-\Delta_1}}{2(1-C)}\frac{ic_1\cos(\frac{\sqrt{\Delta_1}}{2}(x+Vt)) - c_2\sin(-\frac{\sqrt{\Delta_1}}{2}(x+Vt))}{ic_1\sin(\frac{\sqrt{\Delta_1}}{2}(x+Vt)) + c_2\cos(-\frac{\sqrt{\Delta_1}}{2}(x+Vt))})^2}$$

$$-\frac{1}{2}\frac{8bVAC - 8bVA - 2bVB^2 - 1 - V}{a} - \frac{6bV(C^2 - 2C + 1)}{a}$$

$$(\frac{B\sqrt{-\Delta_1}}{2(1-C)}\frac{ic_1\cos(\frac{\sqrt{\Delta_1}}{2}(x+Vt)) - c_2\sin(-\frac{\sqrt{\Delta_1}}{2}(x+Vt))}{ic_1\sin(\frac{\sqrt{\Delta_1}}{2}(x+Vt)) + c_2\cos(-\frac{\sqrt{\Delta_1}}{2}(x+Vt))})^2$$

(3). When $B = 0$ and $\Delta_2 = A(C-1) \geq 0$,then

$$u_{43}(x,t) = -\frac{3}{8}\frac{bV(16A^2C^2 - 32A^2C + 16A^2}{a(C^2 - 2C + 1)}\frac{1}{(\frac{\sqrt{\Delta_2}}{(1-C)}\frac{c_1\cos(\sqrt{\Delta_2}(x+Vt)) + c_2\sin(\sqrt{\Delta_2}(x+Vt))}{c_1\sin(\sqrt{\Delta_2}(x+Vt)) - c_2\cos(\sqrt{\Delta_2}(x+Vt))})^2}$$

$$-\frac{1}{2}\frac{8bVAC - 8bVA - 1 - V}{a} - \frac{6bV(C^2 - 2C + 1)}{a}(\frac{\sqrt{\Delta_2}}{(1-C)}\frac{c_1\cos(\sqrt{\Delta_2}(x+Vt)) + c_2\sin(\sqrt{\Delta_2}(x+Vt))}{c_1\sin(\sqrt{\Delta_2}(x+Vt)) - c_2\cos(\sqrt{\Delta_2}(x+Vt))})^2$$

(4). When $B = 0$ and $\Delta_2 = A(C-1) < 0$,then

$$u_{44}(x,t) = -\frac{3}{8}\frac{bV(16A^2C^2 - 32A^2C + 16A^2}{a(C^2 - 2C + 1)}\frac{1}{(\frac{\sqrt{-\Delta_2}}{(1-C)}\frac{ic_1\cosh(\sqrt{-\Delta_2}(x+Vt)) - c_2\sinh(\sqrt{-\Delta_2}(x+Vt)}{ic_1\sinh(\sqrt{-\Delta_2}(x+Vt)) - c_2\cosh(\sqrt{-\Delta_2}(x+Vt))})^2}$$

$$-\frac{1}{2}\frac{8bVAC - 8bVA - 1 - V}{a} - \frac{6bV(C^2 - 2C + 1)}{a}(\frac{\sqrt{-\Delta_2}}{(1-C)}\frac{ic_1\cosh(\sqrt{-\Delta_2}(x+Vt)) - c_2\sinh(\sqrt{-\Delta_2}(x+Vt))}{ic_1\sinh(\sqrt{-\Delta_2}(x+Vt)) - c_2\cosh(\sqrt{-\Delta_2}(x+Vt))})^2$$

4 Summary and conclusion

In summary,the improved (G'/G) -expansion method with symbolic computation is developed to deal with the nonlinear ZKBBM equation.When applying the proposed method to construct the exact interaction soliton solutions of the nonlinear ZKBBM equation,we get a rich variety of exact solutions which include exponential function solutions, hyperbolic function solutions and triangular function solutions. Further more, our method can obtain more types of travelling solutions mentioned above. We also see that our method is different from the old (G'/G) -expansion method.We use the new auxiliary ordinary differential equations to construct more types of travelling solutions.Our method is more powerful and much easier to solve nonlinear evolution equations. We believe that this method should play an important role in finding exact solutions of NLPDEs.

Note that the nonlinear evolution equations proposed in the present paper are difficult and more general. Therefore, the solutions of the proposed nonlinear evolution equation in this paper have many potential applications in physics.

Acknowledgement

This research was supported by the Natural Science Foundation of Shandong Province of China under Grant (ZR2010AM014) and the Scientific Research Fund from Shandong Provincial Education Department (J07WH04).

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